HAMILTONIAN APPROACH TO QCD_2 IN THE AXIAL GAUGE.

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The Hamiltonian approach is developed for QCD_2 in the limit of infinite number of colours N_C ('t Hooft model). Bosonization of the theory is performed explicitly and the generalized Bogoliubov transformation for the composite boson operators is introduced and used to bring the Hamiltonian into diagonal form in the two-body sector. The resulting theory is re-formulated in terms of effective degrees of freedom and describes the free mesons with creation and annihilation operators commuting in the canonical way. Corrections in N_C to the leading term in the Hamiltonian describe the interaction between mesons and can be used to consider their decays. Chiral properties of the theory are discussed and it is shown that the backward motion of the chiral pion (Goldstone mode) is not suppressed and should contribute to the decays amplitudes.

In 1974 a model for QCD in two dimensions in the limit of infinite number of colours was suggested by 't Hooft [1]. It is described by the Lagrangian

$$L(x) = -\frac{1}{4}F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) + \bar{q}(x)(i\hat{D} - m)q(x), \tag{1}$$

where $\hat{D} = (\partial_\mu - igA^a_\mu t^a)\gamma_\mu$, and the convention for γ-matrices is $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_2$, $\gamma_5 = \gamma_0\gamma_1$. The large N_C limit implies that $g^2 N_C$ remains finite.

For 25 years this model has been under intent attention of many theorists since it possesses many features similar to those of four dimensional QCD and most of them can be studied analytically. This model is a brilliant test-bed for investigating such nonperturbative phenomena as confinement and chiral symmetry breaking.

The fact that two-dimensional gluon contains no propagating degrees of freedom allows to construct a self-consistent Hamiltonian approach. According to the standard rules one finds the Hamiltonian of the model in the axial gauge $A_1 = 0$ to be

$$H = \int dx q^+(x)\left(-i\gamma_5\frac{\partial}{\partial x} + m\gamma_0\right)q(x)$$

$$-\frac{g^2}{2}\int dxdy\; q^+(x)t^a q(x) q^+(y) t^a q(y)\frac{|x-y|}{2}, \tag{2}$$

where "dressed" quark field defined by means of the standard Boguliubov-Valatin angle θ is introduced:

$$q_i(x_0, x) = \int \frac{dk}{2\pi}\left[u(k)b_i(x_0,k) + v(-k)d_i(x_0,-k)\right]e^{ikx} \tag{3}$$

$$u(k) = T(k)\begin{pmatrix}1\\0\end{pmatrix} \quad v(-k) = T(k)\begin{pmatrix}0\\1\end{pmatrix} \quad T(k) = e^{-\frac{1}{2}\theta(k)\gamma_1}.$$

First we shall recall the diagonalization of the model in the one-body sector to find the exact form of the "dressed" quarks and to define the "true" quark vacuum of the model which will appear chirally non-symmetric.

After arranging the normal ordering in Hamiltonian (2) in the given basis (3) one has the Hamiltonian consisting of three parts: vacuum energy, the term quadratic in quark operators ($:H_2:$) and the one of the fourth power ($:H_4:$).

The standard requirement that H_2 be diagonal in terms of quark creation and annihilation operators leads to the gap equation[2]

$$p\cos\theta(p) - m\sin\theta(p) = \frac{\gamma}{2}\,⨍\frac{dk}{(p-k)^2}\sin[\theta(p)-\theta(k)], \tag{4}$$

or equivalently to the system of two coupled equations where the "dressed" quark dispersive law $E(p)$ is introduced[2]:

$$\left\{\begin{array}{l} E(p)\cos\theta(p) = m + \frac{\gamma}{2}\,⨍\frac{dk}{(p-k)^2}\cos\theta(k) \\ \\ E(p)\sin\theta(p) = p + \frac{\gamma}{2}\,⨍\frac{dk}{(p-k)^2}\sin\theta(k), \end{array}\right. \tag{5}$$

It was shown numerically[3] that even in the chiral limit gap equation (4) has a nontrivial solution for θ which gives stable chirally noninvariant vacuum state. We shall return to this issue later when discussing the chiral properties of the model.

It is easy to check that contribution of the H_4 term on the "dressed" quarks states is suppressed by powers of N_C. Thus the total Hamiltonian is completely diagonalized in the one-body sector. Having stopped at this step one can develop a diagrammatic technique with "dressed" quarks lines and Green's functions involved[2], whereas we shall proceed further to perform diagonalization of the model in the mesonic sector.

From the pioneer work by 't Hooft we know that the spectrum of the model consists of mesons, so that it would be very natural to express the Hamiltonian of the theory in terms of mesonic states.

First of all we introduce the two-body operators

$$\begin{array}{ll} B(p,p') = \frac{1}{\sqrt{N_C}}b_i^+(p)b_i(p') & D(p,p') = \frac{1}{\sqrt{N_C}}d_i^+(-p)d_i(-p') \\ \\ M(p,p') = \frac{1}{\sqrt{N_C}}d_i(-p)b_i(p') & M^+(p,p') = \frac{1}{\sqrt{N_C}}b_i^+(p')d_i^+(-p) \end{array} \tag{6}$$

and express the Hamiltonian in their terms. The commutation relations for the new operators can be easily found and the only non-vanishing one reads as follows

$$[M(p,p')M^+(q,q')] \underset{N_C\to\infty}{\to} (2\pi)^2\delta(p'-q')\delta(p-q). \tag{7}$$

As soon as in the mesonic sector of the theory one deals with the $q\bar{q}$ pairs only, so neither isolated quark nor isolated antiquark can be created or annihilated. Such way operators B and D from (6) can not be independent and have to be related to operators M and M^+ somehow. Indeed, it is easy to check that the following substitution[6]

$$\begin{aligned} B(p,p') &= \frac{1}{\sqrt{N_C}}\int\frac{dq}{2\pi}M^+(q,p)M(q,p') \\ D(p,p') &= \frac{1}{\sqrt{N_C}}\int\frac{dq}{2\pi}M^+(p,q)M(p',q) \end{aligned} \tag{8}$$

comes through the commutation relations (7) so that the Hamiltonian takes the form

$$\begin{aligned} H = LN_C\mathcal{E}_v + \int\frac{dQdp}{(2\pi)^2}\Big[&(E(p)+E(Q-p))M^+(p-Q,p)M(p-Q,p) \\ &-\frac{\gamma}{2}\int\frac{dk}{(p-k)^2}\Big\{2C(p,k,Q)M^+(p-Q,p)M(k-Q,k) \\ &+S(p,k,Q)\big(M(p,p-Q)M(k-Q,k)+M^+(p,p-Q)M^+(k-Q,k)\big)\Big\}\Big], \end{aligned} \tag{9}$$

where

$$\begin{aligned} C(p,k,Q) &= \cos\frac{\theta(p)-\theta(k)}{2}\cos\frac{\theta(Q-p)-\theta(Q-k)}{2} \\ S(p,k,Q) &= \sin\frac{\theta(p)-\theta(k)}{2}\sin\frac{\theta(Q-p)-\theta(Q-k)}{2}. \end{aligned} \tag{10}$$

Note that, being of the same order in powers of N_C, H_2 and H_4 parts are equally important in the two-body sector.

As a nest step let us define mesonic creation and annihilation operators in the form[4]

$$\begin{aligned} m_n^+(Q) &= \int\frac{dq}{2\pi}\left\{M^+(q-Q,q)\varphi_+^n(q,Q)+M(q,q-Q)\varphi_-^n(q,Q)\right\} \\ m_n(Q) &= \int\frac{dq}{2\pi}\left\{M(q-Q,q)\varphi_+^n(q,Q)+M^+(q,q-Q)\varphi_-^n(q,Q)\right\}, \end{aligned} \tag{11}$$

where the subscript n numerates mesonic states, and Q being the total momentum of the meson. The given transformation is nothing but another Bogoliubov one generalized for the case of composite bosonic operators. Wave functions φ_+ and φ_- obey the following completeness and orthogonality conditions

$$\begin{aligned}
&\int \frac{dp}{2\pi}\left(\varphi_+^n(p,Q)\varphi_+^m(p,Q) - \varphi_-^n(p,Q)\varphi_-^m(p,Q)\right) = \delta_{nm} \\
&\int \frac{dp}{2\pi}\left(\varphi_+^n(p,Q)\varphi_-^m(p,Q) - \varphi_-^n(p,Q)\varphi_+^m(p,Q)\right) = 0
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\varphi_+^n(p,Q)\varphi_+^n(k,Q) - \varphi_-^n(p,Q)\varphi_-^n(k,Q)\right) = 2\pi\delta\left(p-k\right) \\
&\sum_{n=0}^{\infty}\left(\varphi_+^n(p,Q)\varphi_-^n(k,Q) - \varphi_-^n(p,Q)\varphi_+^n(k,Q)\right) = 0,
\end{aligned} \tag{13}$$

that ensures the standard bosonic commutation relation for the operators m and m^+:

$$\begin{aligned}
&[m_n(Q)m_m^+(Q')] = 2\pi\delta(Q-Q')\ \delta_{nm} \\
&[m_n(Q)m_m(Q')] = [m_n^+(Q)m_m^+(Q')] = 0.
\end{aligned} \tag{14}$$

Particular attention should be payed to the sign "minus" between the two parts of relations (12), (13), which is the direct analogue of the corresponding sign in the standard bosonic Bogoliubov condition $u^2 - v^2 = 1$.

The Hamiltonian of the models takes the final diagonal form in the given basis with $\mathcal{E}_v$ being the vacuum energy density and L — volume of the one-dimensional x-space

$$H = LN_C\mathcal{E}_v + \sum_{n=0}^{+\infty}\int\frac{dQ}{2\pi}Q_n^0(Q)m_n^+(Q)m_n(Q) + O\left(\frac{1}{\sqrt{N_C}}\right), \tag{15}$$

if the wave functions obey the following integral system of equations[2]

$$\left\{\begin{aligned}
&[E(p)+E(Q-p)-Q_0]\varphi_+(p,Q) \\
&\qquad = \gamma \fint \frac{dk}{(p-k)^2}\left[C(p,k,Q)\varphi_+(k,Q) - S(p,k,Q)\varphi_-(k,Q)\right] \\
&[E(p)+E(Q-p)+Q_0]\varphi_-(p,Q) \\
&\qquad = \gamma \fint \frac{dk}{(p-k)^2}\left[C(p,k,Q)\varphi_-(k,Q) - S(p,k,Q)\varphi_+(k,Q)\right].
\end{aligned}\right. \tag{16}$$

A comment on the physical meaning of the two wave functions φ_+ and φ_- is in order: φ_+ describes the forward motion in time of the $q\bar{q}$ pair inside the meson, whereas φ_- stands for the backward motion. Numerical simulations show that φ_- component is suppressed for highly excited mesonic states as well as for rather massive quarks[3].

To trace the root of the unusual form of equations (12), (13) let us re-write the system (16) in the Shrödinger-like matrix form

$$Q_0^n \begin{pmatrix} \varphi_+^n \\ \varphi_-^n \end{pmatrix} = \hat{\mathcal{H}} \begin{pmatrix} \varphi_+^n \\ \varphi_-^n \end{pmatrix} \tag{17}$$

and note that matrix Hamiltonian $\hat{\mathcal{H}}$ appears non-Hermitian in the full Hilbert space of arbitrary functions φ_+ and φ_-. Nevertheless this does not lead to a disaster as "physical" wave functions belong to a restricted space defined by means of projectors $\Lambda_\pm = T(k)\frac{1\pm\gamma_0}{2}T^+(k)$ and one can check explicitly that all eigenenergies of the system (16) are real[4]. Distorted norm (12) is nothing but another reflection of the above space restriction.

Let us discuss possible applications of the developed method. We shall concentrate on the chiral properties of the model.

The general idea of calculation of any matrix element between hadronic states is to express the corresponding operator in terms of mesonic creation and annihilation operators m and m^+ and to calculate the matrix element explicitly using the standard second quantization technique.

Chiral condensate is the first example. First of all, note that Bogoliubov transformation not only changes operators but also re-orders the vacuum state. Luckily the contribution of this re-ordering has sub-leading order in N_C when a matrix element of any bilinear combination of quarks fields is calculated. Thus following the way described above one arrives at

$$\langle\bar{q}q\rangle = -\frac{N_c}{2\pi}\int\limits_{-\infty}^{+\infty} dk\cos\theta(k). \tag{18}$$

Substituting the numerical solution for θ into (18) one finds nonzero result (as expected for chirally non-invariant vacuum) which is in agreement with the one found from the sum rules[5].

Despite of the fact that the bound state equation (16) is rather a subject to numerical study, an appropriate solution for the lowest massless state can be found analytically in the chiral limit

$$\varphi_\pm^\pi(p,Q) = \sqrt{\frac{\pi}{2Q}}\left(\cos\frac{\theta(Q-p)-\theta(p)}{2} \pm \sin\frac{\theta(Q-p)+\theta(p)}{2}\right) \tag{19}$$

and this solution is nothing but the chiral pion — Goldstone mode present in the spectrum of the model. The two comments are in order here: i) being massless this state is not defined in the rest frame (at $Q\to 0$) as clearly seen from equation (19); ii) for finite Q's both wave functions φ_+^π and φ_-^π are of comparable order of magnitude, so that the $q\bar{q}$ pair spends about half time moving forward and about halt time moving backward in time. Thus in contrary to the higher excited states, for which φ_- component is suppressed, in case of pion both wave functions are of the same importance and none of them can be neglected.

Armed with the π-meson solution one can find the pionic decay constant f_π defined as

$$\langle\Omega\,|\,J_\mu^5(x)\,|\,\pi(Q)\,\rangle = f_\pi Q_\mu \frac{e^{-iQx}}{\sqrt{2Q_0}} \tag{20}$$

to be $\sqrt{N_C/\pi}$. Besides the celebrated Gell-Mann-Oakes-Renner relation

$$f_\pi^2 M_\pi^2 = -2m\langle\bar{q}q\rangle, \tag{21}$$

can be readily reconstructed from the system (16).

In conclusion we would like to note that the applications of the developed approach are not exhausted with the given examples. Mesonic decays and scattering can be studied if the next-to-leading terms in the Hamiltonian of the theory are taken into account. This might help, for example, to disclose the role played by pions, as massless Goldstone modes, in the hadronic processes.

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